The Structure of a Real Linear Combination of Two Projections

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ABSTRACT

A necessary and sufficient condition for a Hermitian operator on a Hilbert space to be expressed as $aP + bQ$ with some projections P, Q and reals a, b is established. Moreover the spectrum of such an operator is explicitly drawn in terms of the spectrum of $PQ|\operatorname{ran} P$, the restriction to the range of P.

1. INTRODUCTION

In this paper we study the structure of a real linear combination of two projections on a Hilbert space. Here a projection means a Hermitian idempotent operator.

Continuing the study in a previous paper [8] for the case of matrices, we shall give a necessary and sufficient condition for a Hermitian operator to be expressed as a linear combination of two projections. Such a condition seems to be lacking so far, except that Fillmore [5] obtained a condition for the representability as a sum of two projections, and Davis [2] obtained one as a difference of two projections.

Throughout the paper operators mean bound linear operators on a Hilbert space $\mathscr{H}$. The symbols I, 0 stand for the identity and null operator respectively, while P and Q denote projections with range $\mathscr{M}$ and $\mathscr{N}$ respectively. $\mathscr{M}^{\perp}$ always denotes the orthogonal complement of $\mathscr{M}$, and $P^{\perp} = I - P$; accordingly $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$ $(= \operatorname{ran} P \oplus \operatorname{ran} P^{\perp})$, where $\oplus$ denotes mutually orthogonal direct sum of two subspaces.

A Hermitian operator A is said to be positive (nonnegative) and denoted by $0 < A$ $(0 \leqslant A)$ if $0 < \langle Ax, x\rangle$ $(0 \leqslant \langle Ax, x\rangle)$ for any nonzero vector x, where $\langle\ ,\ \rangle$ denotes the inner product of the Hilbert space $\mathscr{H}$. Order relations among Hermitian operators always refer to these notions, that is, $A < B$ means $B - A$ is positive, and so on.

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2. PROJECTIONS IN GENERIC POSITION

The relation between two projections (equivalently, two subspaces) and relevant theories have been investigated by many authors (e.g., [2], [3], [6]).

The following identity has been well used:

$$(P^{\perp} - Q)^2 = I - (P - Q)^2, \tag{1}$$

while the identities listed below, which are used later, seem not to have been explicitly mentioned anywhere:

$$PQ - QP = QP^{\perp} - P^{\perp}Q = Q^{\perp}P - PQ^{\perp} = P^{\perp}Q^{\perp} - Q^{\perp}P^{\perp}, \tag{2}$$

$$P - Q = PQ^{\perp} - P^{\perp}Q, \tag{3}$$

$$PQ^{\perp} + QP^{\perp} = PQ^{\perp}P + P^{\perp}QP^{\perp}, \tag{4}$$

$$P(PQ - QP) = (PQ - QP)P^{\perp} \quad \text{and} \quad Q(PQ - QP) = (PQ - QP)Q^{\perp}. \tag{5}$$

The mutually orthogonal subspaces $\mathscr{M} \cap \mathscr{N}$, $\mathscr{M} \cap \mathscr{N}^{\perp}$, $\mathscr{M}^{\perp} \cap \mathscr{N}$, and $\mathscr{M}^{\perp} \cap \mathscr{N}^{\perp}$ are all reducing for both P and Q (hence also $aP + bQ$), and the restriction of $aP + bQ$ to each of these subspaces is a scalar. Therefore, in studying the structure of $aP + bQ$, the restrictions to these subspaces are of no interest. In this respect our concern will be only with the case in which

$$\mathscr{M} \cap \mathscr{N} = \mathscr{M} \cap \mathscr{N}^{\perp} = \mathscr{M}^{\perp} \cap \mathscr{N} = \mathscr{M}^{\perp} \cap \mathscr{N}^{\perp} = \{0\}. \tag{6}$$

Following Halmos [6], let us say that projections P, Q are *in generic position* if their ranges $\mathscr{M} = \operatorname{ran} P$ and $\mathscr{N} = \operatorname{ran} Q$ satisfy (6). Dixmier [3] and Davis [2] used the term "position p."

Halmos [6] gave a characterization of projections in generic position. We present here another characterization in the form useful to our aim.

LEMMA 1. *The following three conditions are mutually equivalent:*

(a) *P and Q are in generic position.*
(b) *$PQ - QP$ is quasiinvertible (i.e., one to one and with dense range).*
(c) *$0 < (P - Q)^2 < I$.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) is derived immediately from the identity

$$\ker(PQ - QP) = (\mathcal{M} \cap \mathcal{N}) \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}) \oplus (\mathcal{M} \cap \mathcal{N}^{\perp}) \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp}), \tag{7}$$

because $[\operatorname{ran}(PQ - QP)]^{\perp} = \ker(PQ - QP)$. To show the identity (7), it suffices to prove that the left hand side is contained in the right, because the reverse inclusion is clear. To this end let $x \in \ker(PQ - QP)$ and decompose x as $x = PQx + PQ^{\perp}x + P^{\perp}Qx + P^{\perp}Q^{\perp}x$. Then by (2), the equalities $PQx = QPx$, $PQ^{\perp}x = Q^{\perp}Px$, $P^{\perp}Qx = QP^{\perp}x$, and $P^{\perp}Q^{\perp}x = Q^{\perp}P^{\perp}x$ hold. These equalities lead to $PQx \in \mathcal{M} \cap \mathcal{N}$, $PQ^{\perp}x \in \mathcal{M} \cap \mathcal{N}^{\perp}$, $P^{\perp}Qx \in \mathcal{M}^{\perp} \cap \mathcal{N}$, and $P^{\perp}Q^{\perp}x \in \mathcal{M}^{\perp} \cap \mathcal{N}^{\perp}$, from which (7) follows.

Also the equivalence (a) $\Leftrightarrow$ (c) follows from the following identities together with (1):

$$\ker(P - Q) = (\mathcal{M} \cap \mathcal{N}) \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp}), \tag{8}$$

$$\ker(P^{\perp} - Q) = (\mathcal{M} \cap \mathcal{N}^{\perp}) \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}). \tag{9}$$

These identities (8) and (9) are easily derived from (3) and its variant. This completes the proof. ∎

The following well-known fact (see [4], [7]) will be used in the sequel.

LEMMA 2. *Let A_1 and A_2 be normal operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. If there is a quasiinvertible operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ satisfying $A_2T = TA_1$, then A_1 and A_2 are unitarily equivalent.*

We shall write $A_1 \simeq A_2$ to denote that A_1 and A_2 are unitarily equivalent.

3. MAIN THEOREMS

THEOREM 1. *A Hermitian operator A can be expressed as $A = aP + bQ$ with projections P, Q in generic position and nonzero real a, b if and only if for some real c the following conditions are satisfied:*

(i) *$A - cI$ and $-(A - cI)$ are unitarily equivalent,*
(ii) *either $0 < |A - cI| < |c|I$ or $|c|I < |A - cI|$ is valid.*

Proof. Suppose that $A = aP + bQ$ with P, Q in generic position and nonzero real a, b. Then by Lemma 1 the operator $T = PQ - QP$ is quasiinvertible, and by (5)

$$\{A - \tfrac{1}{2}(a+b)I\}T = T\{aP^{\perp} + bQ^{\perp} - \tfrac{1}{2}(a+b)I\}$$
$$= T\{\tfrac{1}{2}(a+b)I - A\}.$$

Hence $A - \frac{1}{2}(a+b)I$ and $\frac{1}{2}(a+b)I - A$ are unitarily equivalent by Lemma 2, proving (i) with $c = \frac{1}{2}(a+b)$. To see (ii), let us first prove that

$$\tfrac{1}{4}(|a| - |b|)^2 I < \{A - \tfrac{1}{2}(a+b)I\}^2 < \tfrac{1}{4}(|a| + |b|)^2 I. \tag{10}$$

In fact, direct computation shows that (10) is equivalent to

$$-\tfrac{1}{2}(|ab| - ab)I < ab(P-Q)^2 < \tfrac{1}{2}(|ab| + ab)I,$$

which is, in turn, equivalent to $0 < (P-Q)^2 < I$ when $ab \neq 0$. However, the inequalities in the last form are surely guaranteed by Lemma 1. Now (ii) follows immediately from (10), because $|c| = \frac{1}{2}(|a| + |b|)$ or $= \frac{1}{2}||a| - |b||$ according as $ab > 0$ or < 0.

Suppose conversely that A satisfies conditions (i) and (ii) for some c. Without loss of generality we may assume $c \geqslant 0$, by taking $-A$ in place of A if necessary. In terms of the positive part $(A - cI)_+ = \frac{1}{2}\{(A - cI) + |A - cI|\}$ and the negative part $(A - cI)_- = \frac{1}{2}\{-(A - cI) + |A - cI|\}$, condition (i) means that

$$(A - cI)_+ \simeq (A - cI)_-.$$

Let $\mathscr{K} = [\ker(A - cI)_+]^{\perp}$ and $B = (A - cI)_+|_{\mathscr{K}}$, the restriction to $\mathscr{K}$. Then it follows that

$$A - cI \simeq \begin{bmatrix} B & 0 \\ 0 & -B \end{bmatrix}.$$

In terms of B, condition (ii) with $c \geq 0$ means that

$$0 < B < cI \quad \text{or} \quad cI < B. \tag{11}$$

It remains to prove that (11) implies the existence of projections P, Q in generic position and nonzero real a, b such that

$$\begin{bmatrix} B & 0 \\ 0 & -B \end{bmatrix} = aP + bQ - cI.$$

First let us consider the case $cI < B$. Remark that B is invertible if $c > 0$. Take any real d such that $B < dI$, and define S_1 and S_2 by

$$S_1 = \frac{1}{c+d}(B + dcB^{-1}) \quad \text{and} \quad S_2 = \frac{1}{d-c}(-B + dcB^{-1}).$$

with the convention $cB^{-1} = 0$ for $c = 0$. It is readily seen that

$$(c+d)S_1 + (c-d)S_2 = 2B. \tag{12}$$

Further, $I - S_1^2$ and $I - S_2^2$ are positive, and

$$(c+d)(I - S_1^2)^{1/2} + (c-d)(I - S_2^2)^{1/2} = 0 \tag{13}$$

because

$$(c+d)^2(I - S_1^2) = \{I - (cB^{-1})^2\}(d^2 - B^2) = (d-c)^2(I - S_2^2).$$

Now define P and Q by

$$P = \frac{1}{2}\begin{bmatrix} I + S_1 & (I - S_1^2)^{1/2} \\ (I - S_1^2)^{1/2} & I - S_1 \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{2}\begin{bmatrix} I + S_2 & (I - S_2^2)^{1/2} \\ (I - S_2^2)^{1/2} & I - S_2 \end{bmatrix},$$

and a and b by $a = c + d$ and $b = c - d$. Then P and Q are projections, and (12) and (13) show that

$$aP + bQ - cI = \begin{bmatrix} B & 0 \\ 0 & -B \end{bmatrix}.$$

Further, P and Q are in generic position. In fact, since $c = \frac{1}{2}\big||a| - |b|\big|$ and $d = \frac{1}{2}(|a| + |b|)$ by definition, our assumption $cI < B < dI$ yields (10), which is equivalent to $0 < (P - Q)^2 < I$ because $ab \neq 0$. Now by Lemma 1, P and Q are in generic position.

Finally let us consider the case $0 < B < cI$. Apply the preceding argument to the triple $(B, 0, c)$ instead of (B, c, d). Then

$$\begin{bmatrix} B & 0 \\ 0 & -B \end{bmatrix} = cP - cQ = cP + cQ^{\perp} - cI$$

for some projections P, Q in generic position. Let $a = c$ and $b = c$. Clearly P and $Q^{\perp}$ are in generic position. This completes the proof. ∎

THEOREM 2. *Let P, Q be projections in generic position, and a, b nonzero real. Then $aP + bQ$ is unitarily equivalent to*

$$\begin{bmatrix} \frac{1}{2}(a+b)I_{\mathscr{M}} + C^{1/2} & 0 \\ 0 & \frac{1}{2}(a+b)I_{\mathscr{M}} - C^{1/2} \end{bmatrix},$$

where $\mathscr{M} = \operatorname{ran} P$ and $C = \frac{1}{4}(a-b)^2 I_{\mathscr{M}} + abPQ|_{\mathscr{M}}$.

Proof. In the proof of Theorem 1 we showed that

$$aP + bQ - \tfrac{1}{2}(a+b)I \simeq \begin{bmatrix} B & 0 \\ 0 & -B \end{bmatrix}$$

for some positive operator B on a Hilbert space $\mathscr{K}$. Therefore

$$aP + bQ \simeq \begin{bmatrix} \frac{1}{2}(a+b)I_{\mathscr{K}} + B & 0 \\ 0 & \frac{1}{2}(a+b)I_{\mathscr{K}} - B \end{bmatrix}.$$

It remains to prove that $B \simeq C^{1/2}$. To this end, first note that the following identity is valid:

$$\left\{aP + bQ - \tfrac{1}{2}(a+b)I\right\}^2 = P\left\{\tfrac{1}{4}(a+b)^2 I - abQ^{\perp}\right\}P + P^{\perp}\left\{\tfrac{1}{4}(a+b)^2 I - abQ\right\}P^{\perp}. \tag{14}$$

In fact, direct computation shows

$$\begin{aligned}\left\{aP + bQ - \tfrac{1}{2}(a+b)I\right\}^2 &= -(aP + bQ)(aP^{\perp} + bQ^{\perp}) + \tfrac{1}{4}(a+b)^2 I \\ &= \tfrac{1}{4}(a+b)^2 I - ab(PQ^{\perp} + QP^{\perp}),\end{aligned}$$

and (14) follows from (4). Consider again the operator $T = PQ - QP$ that maps $\mathscr{M} = \operatorname{ran} P$ injectively to a dense set of $\mathscr{M}^{\perp} = \operatorname{ran} P^{\perp}$. Since (5) yields

$$P\left\{\tfrac{1}{4}(a+b)^2 I - abQ^{\perp}\right\}PT = TP^{\perp}\left\{\tfrac{1}{4}(a+b)^2 I - abQ\right\}P^{\perp}, \quad (15)$$

we have by Lemma 2

$$P\left\{\tfrac{1}{4}(a+b)^2 I - abQ^{\perp}\right\}P\big|_{\mathscr{M}} \simeq P^{\perp}\left\{\tfrac{1}{4}(a+b)^2 I - abQ\right\}P^{\perp}\big|_{\mathscr{M}^{\perp}}.$$

Since obviously

$$C = P\left\{\tfrac{1}{4}(a+b)^2 I - abQ^{\perp}\right\}P\big|_{\mathscr{M}},$$

we have

$$\begin{bmatrix} B^2 & 0 \\ 0 & B^2 \end{bmatrix} \simeq \left\{aP + bQ - \tfrac{1}{2}(a+b)I\right\}^2 \simeq \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}.$$

Then the multiplicity theorem (see e.g., [1, Theorem 9.1 and Corollary 9.12]) results in $B^2 \simeq C$. Now it is easy to see that this is only possible when $B \simeq C^{1/2}$. This completes the proof. ∎

COROLLARY 3. *If P and Q are projections in generic position, then for any real a, b*

$$\sigma(aP + bQ) = \left\{\tfrac{1}{2}(a+b) \pm \tfrac{1}{2}\sqrt{(a-b)^2 + 4ab\mu}\ ;\ \mu \in \sigma(PQ|\operatorname{ran} P)\right\},$$

where $\sigma(\cdot)$ denotes the spectrum.

In fact, for nonzero a, b this follows immediately from Theorem 2, while the assertion is almost obvious if $a = 0$ or $b = 0$.

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